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A Characterization of p-Radical Groups

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Let F be an algebraically closed field of characteristic $p > 0$. Let G be a finite group with a Sylow p-subgroup P. We call G p-radical if the induced module $(F_P)^G$ from the trivial FP-module F_P is semi-simple (see [7; 11; 1, VI, 6]). In [6], S. Koshitani showed that if a vertex of S is contained in $\mathrm{Ker}(S)$ (the kernel of S) for each simple FG-module S, then G is p-radical. In this paper, we will provide a characterization of p-radical groups from which the result of Koshitani follows directly.

In order to state our main result, we consider the following property of a simple FG-module S:

(P) There exist a subgroup U of G and a simple FU-module T such that:

(1) $S = T^G$ and some vertex of S is contained in $\mathrm{Ker}(T)$.

(2) $U \cap P^g \in \mathrm{Syl}_p(U)$ for every $g \in G$.

THEOREM. *The finite Group G is p-radical if and only if every simple FG-module satisfies* (P).

In order to prove the theorem, we need a series of preliminary lemmas.

LEMMA 1. *Let H be a normal p- or p'-subgroup of a finite group G. Let B and $\overline{B}$ be p-blocks of G and $\overline{G} = G/H$, respectively, such that $\overline{B} \subseteq B$. If every simple FG-module in B satisfies* (P), *then so does every simple $F\overline{G}$-module in $\overline{B}$.*

Proof. Let $\overline{S}$ be a simple $F\overline{G}$-module associated with $\overline{B}$ deflated from the simple FG-module S. Since S belongs to B, there exist a subgroup U of G, a simple FU-module T, and a vertex D of S such that $S = T^G$, $D \subseteq \mathrm{Ker}\ (T)$, and $U \cap P^g \in \mathrm{Syl}_p(U)$ for every $g \in G$.

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Since $H \subseteq \mathrm{Ker}(S)$, we have $H \subseteq \mathrm{Ker}(T) \subseteq U$. Let $\overline{U} = U/H$ and let $\overline{T}$ be the simple $F\overline{U}$-module deflated from T. It is clear that $\overline{S} = \overline{T}^{\overline{G}}$. By [8; 4, Theorem 7.8 (i) and Lemma 3.4], D is a vertex of T also. Hence DH/H is a vertex of $\overline{T}$ by [3, Lemma 1.3 (c)], and it follows by [8; 4, Lemma 3.4] that DH/H is a vertex of $\overline{S}$ also. Furthermore, since $D \subseteq \mathrm{Ker}(T)$, we have $DH/H \subseteq \mathrm{Ker}(\overline{T})$. In order to complete the proof, it remains to show that $\overline{S}$ satisfies condition (2) of (P).

Let P be a Sylow p-subgroup of G and let $\overline{P} = PH/H$. It is clear that $\overline{P} \in \mathrm{Syl}_p(\overline{G})$. For any $g \in G$, we let $\bar{g}$ denote the image of g under the natural epimorphism of G onto $\overline{G}$. We easily see that $P^g \cap U \subseteq (PH)^g \cap U$ and that the quotient group $\overline{(PH)^g \cap U}$ of $(PH)^g \cap U$ by H satisfies $\overline{(PH)^g \cap U} \subseteq \overline{P}^{\bar{g}} \cap \overline{U}$. Since $P^g \cap U \in \mathrm{Syl}_p(U)$, it follows that $\overline{P}^{\bar{g}} \cap \overline{U} \in \mathrm{Syl}_p(\overline{U})$, as desired.

The proof of the next lemma is inpsired by [9].

LEMMA 2. *If every simple FG-module in the principal p-block B of the finite group G satisfies* (P), *then G is p-solvable.*

Proof. Let P be a Sylow p-subgroup of G. We may assume that $P \neq 1$. Let S be a simple FG-module in B. By assumption, there exist a subgroup U of G, a simple FU-module T, and a vertex Q of S such that $S = T^G$, $Q \subseteq \mathrm{Ker}\ (T)$, and $U \cap P^g \in \mathrm{Syl}_p(U)$ for every $g \in G$.

We now have

$$S_{Z(P)} = (T^G)_{Z(P)} \cong \bigoplus_s \left(T^s_{U^s \cap Z(P)}\right)^{Z(P)},$$

where s runs through a complete set of representatives for double cosets of G relative to U and $Z(P)$. Since $Q \subseteq \mathrm{Ker}(T) \subseteq U$ and since $U^s \cap P \in \mathrm{Syl}_p(U^s)$ for any $s \in G$, we have that $Q^s \subseteq \mathrm{Ker}(T^s) \subseteq U^s$ and that $Q^{sk} \subseteq U^s \cap P$ for some $k \in U^s$. Furthermore, $Q^{sk} \subseteq \mathrm{Ker}(T^s)$ as $k \in U^s$ and $\mathrm{Ker}(T^s)$ is a normal subgroup of U^s. Set $l = sk$. Then, by [5, Corollary 3.6], there exists a p-block b of $Q^l C_G(Q^l)$ such that Q^l is a defect group of b and $b^G = B$. Since B is the principal p-block of G, the third main theorem of Brauer [8; 5, Theorem 6.1] implies that b is the principal p-block of $Q^l C_G(Q^l)$. It follows that Q^l is the unique Sylow p-subgroup of $Q^l C_G(Q^l)$. Therefore, $Z(P) \subseteq C_G(Q^l)$ as $Q^l \subseteq P$ and hence $Z(P) \subseteq Q^l$. So $Z(P) \subseteq U^s$ and since $Q^l \subseteq \mathrm{Ker}(T^s)$, it follows that $U^s \cap Z(P) = Z(P) \subseteq \mathrm{Ker}(T^s)$. This shows that $Z(P) \subseteq \ker(S)$. Now [1, IV, Lemma 4.12(iii)] implies that $Z(P) \subseteq O_{p',p}(G)$. Hence $G/O_{p',p}(G)$ is p-solvable by Lemma 1 and induction.

The following fact is crucial for the proof of the theorem.

LEMMA 3. *Assume that the finite group G is p-solvable. Then if S is any simple FG-module, there exist a subgroup W of G and a simple FW-module N such that $S = N^G$ and* $\dim(N)$ *is a p'-number.*

Proof. See [10, Theorem 3].

We note the following remark concerning the subgroup W and the simple FW-module N of Lemma 3.

Remark. The subgroup W and the simple FW-module N of Lemma 3 are not unique in general (not even up to conjugacy). There is, however, a canonical way due to Isaacs of determining W and N uniquely up to conjugacy (see [4]).

LEMMA 4. *Let G be a finite group. Let S be a simple FG-module and denote by d remove the p'-part of* $\dim(S)$. *Assume that there exist a subgroup W and a simple FW-module N of some p'-dimension such that $S = N^G$. Then, the following are equivalent*:

(1) $\dim(\mathrm{Hom}_{FG}(S,(F_P)^G)) = d$, *for any* $P \in \mathrm{Syl}_p(G)$,

(2) *a vertex of S is contained in* $\mathrm{Ker}(N)$ *and for any Sylow p-subgroup P of G, we have* $W \cap P^g \in \mathrm{Syl}_p(W)$ *for every* $g \in G$.

Proof. Choose a Sylow p-subgroup P of G such that $W \cap P \in \mathrm{Syl}_p(W)$. Next, set $r = \dim(\mathrm{Hom}_{FG}(S,(F_P)^G))$.

For any $t \in G$, we have $|W^t \cap P| \le |W \cap P|$. It follows that $|W^tP| \ge |WP|$. Thus $|WtP| = |W^tP| \ge |WP|$. Therefore, if c is the number of double cosets of G relative to W and P, we have $c \le |G|\ |WP|^{-1} = |G:W|_{p'}$, the p'-part of the index of W in G. Furthermore, $c = |G:W|_{p'}$ if and only if $W \cap P^g \in \mathrm{Syl}_p(W)$ for every $g \in G$.

Next, we have $S_P = (N^G)_P = \bigoplus_s (N^s_{W^s \cap P})^P$, where s runs through a complete set of representatives for double cosets of G relative to W and P. Therefore,

$$\begin{aligned} r &= \dim\big(\mathrm{Hom}_{FP}\,(S_P, F_P)\big) \\ &= \sum_s \dim\Big(\mathrm{Hom}_{FP}\Big(\big(N^s_{W^s \cap P}\big)^P, F_P\Big)\Big) \\ &= \sum_s \dim\Big(\mathrm{Hom}_{F[W^s \cap P]}\big(N^s_{W^s \cap P}, F_{W^s \cap P}\big)\Big). \end{aligned}$$

Thus, $r \le c \cdot \dim(N)$ and it follows by the above that $r \le |G:W|_{p'} \cdot \dim(N)$.

Since $S = N^G$ and $\dim(N)$ is a p'-number, we have $d = |G:W|_{p'} \cdot \dim(N)$. Therefore $r = d$ if and only if $c = |G:W|_{p'}$ and $\dim(\mathrm{Hom}_{F[W^s \cap P]}(N^s_{W^s \cap P}, F_{W^s \cap P})) = \dim(N)$ for every s.

Now, for any $t \in G$, if $\dim(\mathrm{Hom}_{F[W^t \cap P]}(N^t_{W^t \cap P}, F_{W^t \cap P})) = \dim(N)$, then $N^t_{W^t \cap P} \cong \dim(N) \cdot F_{W^t \cap P}$ and hence $W^t \cap P \subseteq \mathrm{Ker}(N^t)$. It follows that $r = d$ if and only if $W \cap P \subseteq \mathrm{Ker}(N)$ and $W \cap P^g \in \mathrm{Syl}_p(W)$ for every $g \in G$.

As $\dim(N)$ is a p'-number and $W \cap P \in \mathrm{Syl}_p(W)$, we have that $W \cap P$ is a vertex of N by [8; 4, Theorem 7.5]. Hence, $W \cap P$ is also a vertex of S (see [8; 4, Lemma 3.4]).

We finally conclude that $r = d$ if and only if a vertex of S is contained in $\mathrm{Ker}(N)$ and $W \cap P^g \in \mathrm{Syl}_p(W)$ for every $g \in G$.

LEMMA 5. *Assume that the finite group G is a p-solvable and let P be a Sylow p-subgroup of G. Let S be a simple FG-module and d be the p'-part of* $\dim(S)$. *Then S appears d-times as an irreducible constituent of $(F_P)^G$.*

Proof. This follows from [11, Lemma 2; 2, Theorem (2B)].

We are now ready to prove our theorem.

Proof of Theorem. Fix a Sylow p-subgroup P of G. Let S be any simple FG-module and let d be the p'-part of $\dim(S)$.

Assume first that G is a p-radical. Then G is p-solvable by [9, Theorem 1]. By Lemma 3, there exist a subgroup U of G and a simple FU-module T such that $S = T^G$ and $\dim(T)$ is a p'-number. Since G is p-radical, Lemma 5 implies that $\dim(\mathrm{Hom}_{FG}(S,(F_P)^G)) = d$. It follows by Lemma 4 that a vertex of S is contained in $\mathrm{Ker}(T)$ and that $U \cap P^g \in \mathrm{Syl}_p(U)$ for every $g \in G$.

Next assume that every simple FG-module satisfies (P). Then G is p-solvable by Lemma 2. By assumption, there exist a subgroup V of G and a simple FV-module M such that $V \cap P^g \in \mathrm{Syl}_p(V)$ for every $g \in G$, $S = M^G$, and some vertex D of S is contained in $\mathrm{Ker}(M)$. Furthermore, we may assume that $D \subseteq P$. Since G is p-solvable, V is p-solvable and so by Lemma 3, there exist a subgroup W of V and a simple FW-module N such that $M = N^V$ and $\dim(N)$ is a p'-number. Next, let $h \in G$. Since both $V \cap P$ and $V \cap P^h$ are Sylow p-subgroups of V, we can find $k \in V$ such that $V \cap P^h = (V \cap P)^k$. As $D \subseteq \mathrm{Ker}(M)$, we have $D^k \subseteq \mathrm{Ker}(M)$. Moreover, since $\mathrm{Ker}(M) \subseteq \mathrm{Ker}(N)$, we conclude that $D^k \subseteq \mathrm{Ker}(N)$. Hence

$$D^k \subseteq \mathrm{Ker}(N) \cap (V \cap P)^k = \mathrm{Ker}(N) \cap V \cap P^h \subseteq W \cap V \cap P^h = W \cap P^h.$$

Since $S = N^G$ and since $D \subseteq \mathrm{Ker}(N)$, we have that D is a vertex of N by [8; 4, Theorem 7.8(i) and Lemma 3.4]. It follows by [8; 4, Theorem 7.5] that D is a Sylow p-subgroup of W as $\dim(N)$ is a p'-number. Therefore D^k is a Sylow p-subgroup of W also. This forces $W \cap P^h$ to be a Sylow p-subgroup of W.

We have shown above that $D \subseteq \mathrm{Ker}(N)$ and that $W \cap P^h \in \mathrm{Syl}_p(W)$ for every $h \in G$. We may now apply Lemma 4 and conclude that

$\dim(\mathrm{Hom}_{FG}(S,(F_P)^G)) = d$. Now, $\mathrm{Hom}_{FG}(S,(F_P)^G) = \mathrm{Hom}_{FG}(S, \mathrm{soc}(F_P)^G)$. Hence $\dim(\mathrm{Hom}_{FG}(S, \mathrm{soc}(F_P)^G)) = d$ and it follows that S appears d-times as an irreducible direct summand of the semi-simple FG-module $\mathrm{soc}(F_P)^G$. This, combined with Lemma 5, implies that $(F_P)^G$ is semi-simple, as S is arbitrary. The proof of the theorem is now complete.

As mentioned in the introduction, we deduce

COROLLARY 1 [6, Theorem]. *If a vertex of S is contained in* $\mathrm{Ker}(S)$ *for every simple FG-module S, then G is p-radical.*

Proof. This follows from the Theorem, by taking $U = G$ and $T = S$ for any simple FG-module S.

As a consequence of the proof of the "necessary" part of the theorem, we have

COROLLARY 2. *Let G be a p-radical group and let P be a Sylow p-subgroup of G. Then, for any simple FG-module S, there exist a subgroup U of G and a simple FU-module T of p'-dimension such that $S = T^G$, a vertex D of S is contained in* $\mathrm{Ker}(T)$, *and* $U \cap P^g \in \mathrm{Syl}_p(U)$ *for any $g \in G$.*

According to the Remark, the subgroup U and the simple FU-module T above can be uniquely determined up to conjugacy. With that in mind, it is possible to determine the Green correspondents of simple modules for p-radical groups, as the next result shows.

COROLLARY 3. *Let G be p-radical. Let S, U, T,* and *D be as in Corollary* 2. *Then, $(T_{N(D)\cap U})^{N(D)}$ is the $(G, D, N(D))$-Green correspondent of S, where $N(D)$ is the normalizer of D in G.*

Proof. Set $\hat{S} = (T_{N(D)\cap U})^{N(D)}$. Since $D \subseteq \mathrm{Ker}(T)$ and since D is a normal subgroup of $N(D)$, we have that $D \subseteq \mathrm{Ker}(\hat{S})$ by Mackey's decomposition. As $\hat{S}$ is a direct summand of the $FN(D)$-module $S_{N(D)}$ and S has D as a vertex, it follows by [8; 4, Theorem 7.8(i) and Lemma 3.4] that D is a vertex of every indecomposable direct summand of $\hat{S}$. Green's theorem [8; 4, Theorem 4.3] now implies that $\hat{S}$ is indecomposable and that it is the $(G, D, N(D))$-Green correspondent of S, as required.

We conclude with the following example which shows that Koshitani's condition is not necessary.

EXAMPLE. Let A be the additive goup of the field GF(9) of 9 elements, let M be the multiplicative group of GF(9), and let L be the Galois group of GF(9) over GF(3). Then, L is cyclic of order 2, while M is cyclic of order 8, and A is elementary abelian of order 9. Let G be the triple semi-direct product $(A \rtimes M) \rtimes L$, and let the prime p be 2. Then G has two simple FG-modules, the trivial module F_G and a module S of dimension 8 lying over the single G-conjugacy class of all non-trivial linear F-representations of A. By [11, Proposition 2], G is 2-radical. However, G does not satisfy Koshitani's condition, since S is faithful, but has L as a vertex.

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REFERENCES

1. W. Feit, "The Representation Theory of Finite Groups," North-Holland, Amsterdam, 1982.
2. P. Fong, Solvable groups and modular representation theory, *Trans. Amer. Math. Soc.* **103** (1962), 484–494.
3. W. Hamernik and G. Michler, On vertices of simple modules in p-solvable groups, *Mitt. Math. Sem. Giessen* **121** (1976), 147–162.
4. I. M. Isaacs, Characters of π-separable groups, *J. Algebra* **86** (1984), 98–128.
5. R. Knörr, On the vertices of irreducible modules, *Ann. of Math.* (2) **110** (1979), 487–499.
6. S. Koshitani, A remark on p-radical groups, *J. Algebra* **134** (1990), 491–496.
7. K. Motose and Y. Ninomiya, On the subgroups H of a group G such that $J(KH)KG \supset J(KG)$, *Math. J. Okayama Univ.* **17** (1975), 171–176.
8. H. Nagao and Y. Tsushima, "Representations of Finite Groups," Academic Press, London/New York, 1989.
9. T. Okuyama, p-Radical groups are p-solvable, *Osaka J. Math.* **23** (1986), 467–469.
10. Y. Tsushima, On the second reduction theorem of P. Fong, *Kumamoto J. Math.* **13** (1978), 6–14.
11. Y. Tsushima, On p-radical groups, *J. Algebra* **103** (1986), 80–86.